$+ \quad - \quad \times \quad \div$

Complex Arithmetic Operations & Square Roots $\sqrt{Z}$

ch 1.1, 1.2

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D | Imaginary Unit i

$$i^2 = -1 \rightarrow i = \sqrt{-1}$$

note $i^n \in \{i, -1, -i, 1\}$

D | Complex Number

$$z \in \mathbb{C}$$

$$\underbrace{\alpha}_{\text{real part}} + \underbrace{i\beta}_{\text{imaginary part}} \qquad \alpha, \beta \in \mathbb{R}$$

if $\alpha = 0$, z is said to be purely imaginary

notice how $\mathbb{C}$ is formulated via $\mathbb{R}$, this is important for proofs later

Fundamental Operations

addition $(+)$

$$(\alpha + i\beta) + (\gamma + i\delta)$$
$$= (\alpha + \gamma) + i(\beta + \delta)$$

subtraction $(-)$

$$(\alpha + i\beta) - (\gamma + i\delta)$$
$$= (\alpha - \gamma) + (\beta - \delta)$$

multiplication $(\cdot)$

$$(\alpha + i\beta)(\gamma + i\delta)$$
$$= (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma)$$

division $(\div)$ ← in practise you can multiply by conj of divisor

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + i\left(\frac{-\alpha\delta + \beta\gamma}{\gamma^2 + \delta^2}\right)$$
$$= \frac{(\alpha\gamma + \beta\delta) + i(\beta\gamma - \alpha\delta)}{\gamma^2 + \delta^2}$$

reciprocal $(\frac{1}{z})$

$$\frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}$$

square root $(\sqrt{z})$

$$\sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right)$$

P | Complex Division

looking for x, y in terms of $\alpha, \beta, \gamma, \delta$

suppose $\frac{\alpha + i\beta}{\gamma + i\delta} = x + iy \quad (\gamma + i\delta \neq 0)$

$$\rightarrow \alpha + i\beta = (x + iy)(\gamma + i\delta)$$
$$\rightarrow \alpha + i\beta = (\gamma x - \delta y) + i(\delta x + \gamma y)$$
$$\rightarrow \begin{cases} \alpha = \gamma x - \delta y \\ \beta = \delta x + \gamma y \end{cases}$$

easier to split into two dependent equations. otherwise it gets messy...

$$\rightarrow \delta y = \gamma x - \alpha$$
$$\rightarrow y = \frac{\gamma x - \alpha}{\delta}$$
$$\rightarrow \beta = \delta x + \gamma\left(\frac{\gamma x - \alpha}{\delta}\right)$$
$$\rightarrow \beta\delta = \delta^2 x + \gamma(\gamma x - \alpha)$$
$$\rightarrow \beta\delta = \delta^2 x + \gamma^2 x - \gamma\alpha$$
$$\rightarrow \beta\delta = x(\delta^2 + \gamma^2) - \gamma\alpha$$
$$\rightarrow x(\delta^2 + \gamma^2) - \gamma\alpha = \beta\delta$$
$$\rightarrow x = \frac{\beta\delta + \gamma\alpha}{\delta^2 + \gamma^2}$$

$$\gamma x = \alpha + \delta y$$
$$\rightarrow x = \frac{\alpha + \delta y}{\gamma}$$
$$\rightarrow \beta = \delta\left(\frac{\alpha + \delta y}{\gamma}\right) + \gamma y$$
$$\rightarrow \beta\gamma = \delta(\alpha + \delta y) + \gamma^2 y$$
$$\rightarrow \beta\gamma = \delta\alpha + \delta^2 y + \gamma^2 y$$
$$\rightarrow \beta\gamma = y(\delta^2 + \gamma^2) + \delta\alpha$$
$$\rightarrow \delta\alpha - \beta\gamma = -y(\delta^2 + \gamma^2)$$
$$\rightarrow \frac{\beta\gamma - \delta\alpha}{(\delta^2 + \gamma^2)} = y$$
$$\rightarrow y = \frac{\beta\gamma - \delta\alpha}{\delta^2 + \gamma^2}$$

2 ways of saying the divisor can't be 0

$$\gamma + i\delta \neq 0,$$
$$\delta^2 + \gamma^2 \neq 0$$

notice how it collapses into real division if $\delta = 0$

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{\beta\delta + \gamma\alpha}{\delta^2 + \gamma^2} + i\left(\frac{\beta\gamma - \delta\alpha}{\delta^2 + \gamma^2}\right)$$

P | Complex Square Root

suppose $(x + iy)^2 = \alpha + i\beta$

find x, y in terms of α, β

$$x^2 - y^2 + 2ixy = \alpha + i\beta$$
$$\rightarrow \begin{cases} x^2 - y^2 = \cancel{\beta}\alpha \\ 2xy = \beta \end{cases}$$

now solve for x, y

$$\alpha^2 + \beta^2 = (x^2 - y^2)^2 + (2xy)^2$$
$$= x^4 - 2x^2y^2 + y^4 + 4x^2y^2$$
$$= x^4 + 2x^2y^2 + y^4$$
$$\alpha^2 + \beta^2 = (x^2 + y^2)^2$$

only 1 root bc $x^2 + y^2 \geq 0$

$$x^2 + y^2 = \sqrt{\alpha^2 + \beta^2}$$

div | div

$$x^2 + (x^2 - \alpha) = \sqrt{\alpha^2 + \beta^2}$$
$$\rightarrow 2x^2 - \alpha = \sqrt{\alpha^2 + \beta^2}$$
$$\rightarrow x^2 = \tfrac{1}{2}\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)$$

$$(y^2 + \alpha) + y^2 = \sqrt{\alpha^2 + \beta^2}$$
$$\rightarrow 2y^2 + \alpha = \sqrt{\alpha^2 + \beta^2}$$
$$\rightarrow y^2 = \tfrac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + \beta^2}\right)$$

$$\rightarrow \alpha + i\beta = \pm\sqrt{\tfrac{1}{2}\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)} \pm i\sqrt{\tfrac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + \beta^2}\right)}$$

$$\rightarrow \alpha + i\beta = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right) \blacksquare$$

note how the roots are symmetric

need to satisfy $2xy = \beta$ (sign)